Linear Algebra I

30/01/2014, Thursday, 9:00-12:00

You are **NOT** allowed to use any type of calculators.

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1 (12+3=15 pts) **Linear equations**

Consider the linear equation

$$\begin{bmatrix} 1 & 3 & 5 & -2 \\ 1 & 4 & 6 & -2 \\ -1 & -1 & -3 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} a \\ b \\ b-2a \end{bmatrix}$$

where a and b are real numbers.

(a) Find all values of a and b for which the equation is consistent. For these values find the general solution of the equation.

(b) Find all values of a and b for which the equation has a unique solution.

REQUIRED KNOWLEDGE: **linear equations, Gauss-elimination, row reduced echelon form.**

SOLUTION:

We begin with performing row operations in order to put the augmented matrix into the row echelon form:

$$\left[\begin{array}{rrrrr} 1 & 3 & 5 & -2 & a \\ 1 & 4 & 6 & -2 & b \\ -1 & -1 & -3 & 2 & b-2a \end{array}\right] \xrightarrow{\textbf{2nd}=-1\times\textbf{1st}+\textbf{2nd}} \left[\begin{array}{rrrrr} 1 & 3 & 5 & -2 & a \\ 0 & 1 & 1 & 0 & b-a \\ -1 & -1 & -3 & 2 & b-2a \end{array}\right]$$

$$\left[\begin{array}{rrrrr} 1 & 3 & 5 & -2 & a \\ 0 & 1 & 1 & 0 & b-a \\ -1 & -1 & -3 & 2 & b-2a \end{array}\right] \xrightarrow{\textbf{3rd}=\textbf{1st}+\textbf{3rd}} \left[\begin{array}{rrrrr} 1 & 3 & 5 & -2 & a \\ 0 & 1 & 1 & 0 & b-a \\ 0 & 2 & 2 & 0 & b-a \end{array}\right]$$

$$\left[\begin{array}{rrrrr} 1 & 3 & 5 & -2 & a \\ 0 & 1 & 1 & 0 & b-a \\ 0 & 2 & 2 & 0 & b-a \end{array}\right] \xrightarrow{\textbf{3rd}=-2\times\textbf{2nd}+\textbf{3rd}} \left[\begin{array}{rrrrr} 1 & 3 & 5 & -2 & a \\ 0 & 1 & 1 & 0 & b-a \\ 0 & 0 & 0 & 0 & a-b \end{array}\right].$$

1a: It is consistent if and only if $a - b = 0$, i.e. $a = b$. For these values, we can proceed with putting the augmented matrix into the row reduced echelon form in order to solve the equations:

$$\left[\begin{array}{rrrrr} 1 & 3 & 5 & -2 & a \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{\textbf{1st}=-3\times\textbf{2nd}+\textbf{1st}} \left[\begin{array}{rrrrr} 1 & 0 & 2 & -2 & a \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

Therefore, x_1 and x_2 are the lead variables and x_3 and x_4 are the free. Then, the solution can be given as

$$x_1 = a - 2x_3 + 2x_4 \quad \text{and} \quad x_2 = -x_3.$$

1b: When $a \neq b$, the equations are inconsistent. When $a = b$, there are two free variables. Therefore, the system has never a unique solution.

Let

$$M = \begin{bmatrix} A & B \\ C & I \end{bmatrix}$$

where all four blocks are $n \times n$ matrices.

(a) Show that M is nonsingular if and only if $A - BC$ is nonsingular.

(b) Let $D = A - BC$. Suppose that D is nonsingular. Show that

$$M^{-1} = \begin{bmatrix} D^{-1} & -D^{-1}B \\ -CD^{-1} & I + CD^{-1}B \end{bmatrix}.$$

REQUIRED KNOWLEDGE: **nonsingular matrices, inverse of a matrix, partitioned matrices.**

SOLUTION:

2a: Let $z \in \mathbb{R}^{2n}$ be a vector such that $Mz = 0$. Partition z as follows:

$$z = \begin{bmatrix} x \\ y \end{bmatrix}$$

where $x, y \in \mathbb{R}^n$. Then, we have

$$0 = Mz = \begin{bmatrix} A & B \\ C & I \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} Ax + By \\ Cx + y \end{bmatrix}.$$

Hence, we get $Ax + By = 0$ and $Cx + y = 0$. From the latter, we get $y = -Cx$ and hence $(A - BC)x = 0$ from the former. Then, we can conclude that M is nonsingular if and only if so is $A - BC$.

2b: Note that

$$\begin{aligned} \begin{bmatrix} A & B \\ C & I \end{bmatrix} \begin{bmatrix} D^{-1} & -D^{-1}B \\ -CD^{-1} & I + CD^{-1}B \end{bmatrix} &= \begin{bmatrix} AD^{-1} - BCD^{-1} & -AD^{-1}B + B + BCD^{-1}B^{-1} \\ CD^{-1} - CD^{-1} & -CD^{-1}B + I + CD^{-1}B \end{bmatrix} \\ &= \begin{bmatrix} (A - BC)D^{-1} & -(A - BC)D^{-1}B + B \\ 0 & I \end{bmatrix} \\ &= \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} \end{aligned}$$

since $D = A - BC$. Consequently, we have

$$M^{-1} = \begin{bmatrix} D^{-1} & -D^{-1}B \\ -CD^{-1} & I + CD^{-1}B \end{bmatrix}.$$

3 (5+5+5=15 pts) **Row and column spaces**

Let $A \in \mathbb{R}^{m \times n}$, $B \in \mathbb{R}^{n \times r}$, and $C = AB$. Show that

(a) the column space of C is a subspace of the column space of A.

(b) the row space of C is a subspace of the row space of B.

(c) $\operatorname{rank}(C) \leqslant \min\{\operatorname{rank}(A), \operatorname{rank}(B)\}$.

Required Knowledge: **column space and rank of a matrix**

Solution:

3a: Let $y \in R(C)$. Then, there exists $x \in \mathbb{R}^r$ such that $y = Cx = ABx$. Hence, we see that $y \in R(A)$. Therefore, $R(C) \subseteq R(A)$.

3b: Note that $C^T = B^T A^T$. By applying the previous result, we get $R(C^T) \subseteq R(B^T)$. In other words, the row space of C is a subspace of the row space of B.

3c: Note that $\operatorname{rank}(C) = \dim(R(C)) = \dim(R(C^T))$. From 3a and 3b, we have

$$\begin{aligned}\operatorname{rank}(C) = \dim(R(C)) \leqslant \dim(R(A)) = \operatorname{rank}(A)\\ \operatorname{rank}(C) = \dim(R(C^T)) \leqslant \dim(R(B^T)) = \operatorname{rank}(B).\end{aligned}$$

By combining these inequalities, we obtain $\operatorname{rank}(C) \leqslant \min\{\operatorname{rank}(A), \operatorname{rank}(B)\}$.

Consider the vector space of $k \times k$ matrices, i.e. $\mathbb{R}^{k\times k}$.

(a) Let $\lambda \in \mathbb{R}$ and

$$V_\lambda = \{A \in \mathbb{R}^{k\times k} \mid \lambda \text{ is an eigenvalue of } A\}.$$

Show that V_λ is *not* a subspace of $\mathbb{R}^{k\times k}$.

(b) Let $x \in \mathbb{R}^k$ be a nonzero vector and

$$V_x = \{A \in \mathbb{R}^{k\times k} \mid x \text{ is an eigenvector of } A\}.$$

Show that V_x is a subspace of $\mathbb{R}^{k\times k}$.

Required Knowledge: **vector spaces, and subspaces.**

Solution:

4a: Suppose that $\lambda \neq 0$. Then, V_λ is closed neither under scalar multiplication nor vector addition. To verify the latter, observe that $0_{k\times k} \notin V_\lambda$. To verify the former, observe that λ is not an eigenvalue of $A + A = 2A$ even if it is an eigenvalue of A. Suppose, now, that $\lambda = 0$. Then, V_0 is closed under scalar multiplication. However, it is not so under vector addition. For instance, one can take $k = 2$, $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, and $B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ and observe that $A, B \in V_0$ but $A + B \notin V_0$.

4b: Note first that the zero (or the identity) matrix belongs to V_x. As such, V_x is a nonempty set. To show that it is closed under scalar multiplication. Let $A \in V_x$ and $\alpha \in \mathbb{R}$. Note that x is also an eigenvector of αA. For the closedness under vector addition, let $A, B \in V_x$ and note that x is an eigenvector of $A + B$.

Consider the matrix

$$M = \begin{bmatrix} 0 & -2 & 1 \\ 1 & 3 & -1 \\ 0 & 0 & \alpha \end{bmatrix}$$

where α is a real number.

(a) Find the determinant of M.

(b) Find all values of α for which M is nonsingular.

(c) Find the eigenvalues of M. [**Hint:** α is an eigenvalue.]

(d) Find all values of α for which M is diagonalizable.

(e) Let $\alpha = 1$. Find a nonsingular matrix T and a diagonal matrix D such that $M = TDT^{-1}$.

Required Knowledge: **determinants, eigenvalues, and diagonalization.**

Solution:

5a: By using cofactor expansion with respect to the last row, we get

$$\det(M) = \alpha \det(\begin{bmatrix} 0 & -2 \\ 1 & 3 \end{bmatrix}) = 2\alpha.$$

5b: A square matrix is nonsingular if and only if its determinant is not zero. Therefore, M is nonsingular if and only if $\alpha \neq 0$.

5c: Note that

$$\det(M - \lambda I) = \det(\begin{bmatrix} -\lambda & -2 & 1 \\ 1 & 3-\lambda & -1 \\ 0 & 0 & \alpha - \lambda \end{bmatrix}) = (\alpha - \lambda)\det(\begin{bmatrix} -\lambda & -2 \\ 1 & 3-\lambda \end{bmatrix}) = (\alpha - \lambda)(\lambda^2 - 3\lambda + 2).$$

Therefore, the eigenvalues can be found as $\lambda_1 = \alpha$, $\lambda_2 = 1$, and $\lambda_3 = 2$.

5d: An $n \times n$ matrix is diagonalizable if and only if it has n linearly independent eigenvalues. In articular, this holds whenever the matrix has distinct eigenvalues. Therefore, we can conclude that M is diagonalizable if $\alpha \notin \{1, 2\}$. Then, we need to check two cases: $\alpha = 1$ and $\alpha = 2$. For the case $\alpha = 1$, we can find the eigenvectors corresponding the eigenvalue $\lambda = 1$ by solving the following linear equation:

$$0 = (M - I)x = \begin{bmatrix} -1 & -2 & 1 \\ 1 & 2 & -1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}.$$

This yields two linearly independent eigenvalues, for instance, $x = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ and $x = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$. As such, one can conclude that M is diagonalizable if $\alpha = 1$. For the case $\alpha = 2$, we can find the eigenvectors corresponding the eigenvalue $\lambda = 2$ by solving the following linear equation:

$$0 = (M - 2I)x = \begin{bmatrix} -2 & -2 & 1 \\ 1 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}.$$

This would yield at most one linearly independent eigenvector, say $x = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$. Hence, we can conclude that M is not diagonalizable whenever $\alpha = 2$. Consequently, M is diagonalizable if and only if $\alpha \neq 2$.

5e: For $\alpha = 1$, we have already found two linearly independent eigenvalues corresponding to the eigenvalue $\lambda = 1$ in the previous problem. For the eigenvalue $\lambda = 2$, we can find an eigenvector by solving:

$$0 = (M - 2I)x = \begin{bmatrix} -2 & -2 & 1 \\ 1 & 1 & -1 \\ 0 & 0 & -1 \end{bmatrix} x.$$

This would yield, for instance, $x = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$. Then, we can take

$$T = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 1 & 2 & 0 \end{bmatrix} \quad \text{and} \quad D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}.$$

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6 (10 pts) **Least squares problem**

Let x_i and y_i be real numbers with $i = 1, 2, \ldots, n$. Suppose that

$$x_1 + x_2 + \cdots + x_n = y_1 + y_2 + \cdots + y_n = 0.$$

Show that the least squares solution of

$$\begin{bmatrix} x_1 & 1 \\ x_2 & 1 \\ \vdots & \vdots \\ x_n & 1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$$

is given by

$$a = \frac{x_1y_1 + x_2y_2 + \cdots + x_ny_n}{x_1^2 + x_2^2 + \cdots + x_n^2} \quad \text{and} \quad b = 0.$$

REQUIRED KNOWLEDGE: **least-squares problem.**

SOLUTION:

Note that the normal equations to solve are given by

$$\begin{bmatrix} x_1 & 1 \\ x_2 & 1 \\ \vdots & \vdots \\ x_n & 1 \end{bmatrix}^T \begin{bmatrix} x_1 & 1 \\ x_2 & 1 \\ \vdots & \vdots \\ x_n & 1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} x_1 & 1 \\ x_2 & 1 \\ \vdots & \vdots \\ x_n & 1 \end{bmatrix}^T \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$$

$$\begin{bmatrix} x_1 & x_2 & \cdots & x_n \\ 1 & 1 & \cdots & 1 \end{bmatrix} \begin{bmatrix} x_1 & 1 \\ x_2 & 1 \\ \vdots & \vdots \\ x_n & 1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} x_1 & x_2 & \cdots & x_n \\ 1 & 1 & \cdots & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$$

$$\begin{bmatrix} x_1^2 + x_2^2 + \cdots + x_n^2 & x_1 + x_2 + \cdots + x_n \\ x_1 + x_2 + \cdots + x_n & n \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} x_1y_1 + x_2y_2 + \cdots + x_ny_n \\ y_1 + y_2 + \cdots + y_n \end{bmatrix}$$

$$\begin{bmatrix} x_1^2 + x_2^2 + \cdots + x_n^2 & 0 \\ 0 & n \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} x_1y_1 + x_2y_2 + \cdots + x_ny_n \\ 0 \end{bmatrix}.$$

Then, we have

$$\begin{aligned} \begin{bmatrix} a \\ b \end{bmatrix} &= \begin{bmatrix} x_1^2 + x_2^2 + \cdots + x_n^2 & 0 \\ 0 & n \end{bmatrix}^{-1} \begin{bmatrix} x_1y_1 + x_2y_2 + \cdots + x_ny_n \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} \frac{1}{x_1^2+x_2^2+\cdots+x_n^2} & 0 \\ 0 & \frac{1}{n} \end{bmatrix} \begin{bmatrix} x_1y_1 + x_2y_2 + \cdots + x_ny_n \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} \frac{x_1y_1+x_2y_2+\cdots+x_ny_n}{x_1^2+x_2^2+\cdots+x_n^2} \\ 0 \end{bmatrix}. \end{aligned}$$